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Conformal quantisation of electrodynamics

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Abstract. A new treatment of Maxwell's equations is proposed. It is shown that these equations can play the role of subsidiary conditions which separate the conformal symmetric solutions of a fixed third-order differential equation. On the basis of this equation a quantisation of Maxwell's equations is constructed such that the conformal symmetry is preserved only in the physical state space. The proposed procedure of conformal quantisation is illustrated by the example of a free electromagnetic field.

1. Introduction

Investigations in conformal-invariant quantum field theory have shown that the conformal group can be used as a method for an exact solution of some field models. For instance in the paper of Aneva *et al* (1981) an exact solution of the quantum massless Thirring model has been constructed using conformal symmetry. In four-dimensional space–time the massless QED is the most interesting theory from the point of view of conformal invariance. However, the present methods of quantisation of the electromagnetic field with strong gauge fixing do not lead to conformal invariant theories even in the case of free fields. On the other hand, the absence of charged massless particles in QED could be explained naturally on the basis of exact conformal symmetry. For example, the full two-point Wightman function of massless fermions, found in Sotkov and Stoyanov (1980) using conformal symmetry, has no pole. It is evident that for a physically reliable result, it is also necessary to know how to quantise Maxwell's equations in a conformally invariant way. As is well known, in a conformal-invariant quantum field theory, using only irreducible linear representations of the conformal group one can only obtain the pure longitudinal photon Wightman function. Baker and Johnson (1979) have suggested that in order to eliminate this difficulty, one should consider conformal invariance up to local gauge transformations (see also in this connection Todorov *et al* (1978)). Another way out of this situation is by the nonlinear realisations of the conformal group, proposed by Sotkov and Stoyanov (1980).

In this paper the new treatment of Maxwell's equations is discussed. The main point of this approach is that these equations play the role of subsidiary conditions which single out the conformal-symmetric solutions of a fixed third-order differential equation. It is shown, on the basis of this equation, that a quantisation of the Maxwell's equations exists such that the conformal symmetry is preserved only in the physical state space. It turned out that the space of the physical photon states is included in the vector field state space, which is larger than the one in the Gupta–Bleuler gauge.

We illustrate the proposed procedure for conformal quantisation by the example of a free electromagnetic field.

Finally, as a natural consequence of our arguments, it has been proved that the photon spaces in Gupta–Bleuler gauge with and without interaction have no common elements.

2. Conformal-invariant gauge condition

We consider the pair $(D(x), J_\mu(x))$, consisting of a scalar field $D(x)$ and the vector field $J_\mu(x)$ with scale dimensions $d_D = 4$ and $d_{J_\mu} = 3$ (in mass units) which have the usual homogeneous dilatation laws and the following special conformal transformations:

$$D'(x) = \frac{1}{(\rho(\alpha, x))^4} D(x') - 4H \frac{\alpha^2(x^\nu + \alpha^\nu x^2) - \alpha^\nu \rho(\alpha, x)}{(\rho(\alpha, x))^4} J_\nu(x) \tag{2.1}$$

$$J'_\mu(x) = (\rho(\alpha, x))^{-2} (\partial x'^\nu / \partial x^\mu) J_\nu(x') \tag{2.2}$$

where $\mu, \nu = 0, 1, 2, 3$; x^μ are the coordinates of a point in Minkowski space-time with metric $g_{\mu\nu}$: diag $g_{\mu\nu} = (1, -\mathbb{1}_3)$; H is a fixed constant; x'_μ denote the coordinates of the point x_μ after special conformal transformations (with parameters α_μ):

$$x'_\mu = \frac{x_\mu + \alpha_\mu x^2}{\rho(\alpha, x)} \qquad \rho(\alpha, x) = 1 + 2(\alpha \cdot x) + \alpha^2 x^2 \qquad \alpha \cdot x = \alpha^\mu x^\nu g_{\mu\nu}.$$

A realisation of the conformal group of this kind has been considered by Bayen and Flato (1976) in connection with the conformal invariance of Maxwell's equations. We shall use these representations of the conformal group to formulate a conformal-invariant gauge condition for the electromagnetic potential $A_\mu(x)$ not only for the case of free fields, but also for interacting fields. For this purpose we shall consider two explicit constructions of the field pair $(D(x), J_\mu(x))$ with transformation laws (2.1) and (2.2):

(a) The first one arises when we identify the field J_μ with a conserved current j_μ, i.e. $J_\mu(x) \equiv j_\mu(x)$ and

$$\partial^\mu j_\mu(x) = 0. \tag{2.3}$$

As is well known, when the current has a canonical dimension $d_{j_\mu} = 3$, the condition (2.3) is invariant with respect to the transformation (2.2). In order to construct the field $D(x)$ we introduce another scalar field $S(x)$ with a dimension $d_S = 0$ and with the following special conformal transformations:

$$S'(x) = S(x') - eq \ln|\rho(\alpha, x)| \tag{2.4}$$

where q is an arbitrary constant and e is a dimensionless electric charge. The field with such transformation properties has been considered in Sotkov and Stoyanov (1980). It can be verified directly that under the conformal group the scalar quantity

$$-(2H/eq)\partial_\mu(S(x) j^\mu(x))$$

transform according to (2.1) and, therefore, it is a concrete realisation of $D(x)$. In this way we obtain the first pair:

$$D(x) = -(2H/eq)\partial^\mu(S j_\mu) \qquad J_\mu(x) = j_\mu. \tag{2.5}$$

(b) The second realisation we find with the help of the electromagnetic vector potential A_μ with a dimension $d_{A_\mu} = 1$ and with the following special conformal transformation (see Sotkov *et al* (1979)):

$$A'_\mu(x) = (\partial x'^\nu/\partial x^\mu)A_\nu(x') + C\partial_\mu \ln|\rho(\alpha, x)|, \tag{2.6}$$

where C is an arbitrary constant. Starting from equation (2.6) it is easy to check that the fields $H\Box\partial^\mu A_\mu$ and $\partial^\nu F_{\nu\mu}$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) transform according to the laws (2.1) and (2.2) respectively. Thus, we obtain the second realisation:

$$D(x) = H\Box\partial^\mu A_\mu \qquad J_\mu(x) = \partial^\nu F_{\nu\mu}. \tag{2.7}$$

We equate the corresponding expressions (2.5) and (2.7) for $D(x)$ and $J_\mu(x)$ and obtain the following system of equations

$$\Box\partial^\mu A_\mu(x) = -(2/eq)j^\mu(x)\partial_\mu S(x) \tag{2.8}$$

$$\partial^\mu F_{\mu\nu}(x) = j_\nu(x) \tag{2.9}$$

which is evidently conformal invariant. Thus, we have proved the statement: the system of differential equations (2.8) and (2.9) is invariant with respect to the conformal tranformation given by (2.2), (2.4) and (2.6).

Equations (2.9) coincide with Maxwell's equation and equation (2.8) can be considered as a gauge condition. We shall note here two important properties of the system obtained above.

Property 1. The conformal symmetric solutions of equation (2.8) satisfy Maxwell's equations (2.9) and *vice versa.*

Here we used the following *definition.* Given equation (2.8) with fixed $S(x)$ and $j_\mu(x)$, let $A_\mu(x)$ be its arbitrary solution. We shall call a solution conformal symmetric if the result $A'_\mu(x)$ obtained after the transformations (2.6) satisfies the equation

$$\Box\partial^\mu A'_\mu(x) = -(2/eq)j'_\mu(x)\partial^\mu S'(x). \tag{2.10}$$

The proof of property 1 follows directly from the conformal invariance of the system equations (2.8) and (2.9). Indeed, substituting A'_μ, S' and j'_μ from (2.2), (2.4) and (2.6) in equation (2.10) for infinitesimal values of the parameters α_μ we obtain the identity

$$\begin{aligned}&[(x^2\delta^\nu_\mu - 2x_\mu x^\nu)\partial_\nu - 8x_\mu]\Box\partial^\rho A_\rho + 4\partial^\rho F_{\rho\mu}\\ &\qquad = -(2/eq)[(x^2\delta^\nu_\mu - 2x_\mu x^\nu)\partial_\nu - 8x_\mu]j^\rho(x)\partial_\rho S(x) + 4j_\mu(x)\end{aligned} \tag{2.11}$$

for which our statement follows immediately.

Thus, we establish the fact that Maxwell's equations can be considered as a subsidiary condition which singles out the conformal symmetric solutions of (2.8).

Property 2. The system of equations (2.8) and (2.9) is preserved by the gauge transformations

$$A_\mu \to A_\mu + \partial_\mu\phi \qquad j_\mu \to j_\mu \qquad S \to S + \text{constant} \tag{2.12}$$

if the field $\phi(x)$ satisfies the equation

$$\Box^2\phi(x) = 0 \qquad \Box = g_{\mu\nu}\partial^\mu\partial^\nu. \tag{2.13}$$

This can be verified directly by substitution of (2.12) in (2.8) and (2.9). The conformal invariance of (2.8) and (2.9) shows that the conformal transformations can change the gauge arbitrariness (2.13). It is easy to see that any restriction on this gauge freedom leads to breakdown of the conformal symmetry. From this point of view we can consider (2.8) as a conformal invariant gauge condition for the electromagnetic potential $A_\mu(x)$.

3. Conformal invariant quantisation of Maxwell's equations

The results of § 2 allow us to formulate a conformal invariant procedure for quantisation of the Maxwell equations:

$$\partial^\mu F_{\mu\nu} \equiv \Box A_\nu - \partial_\nu \partial^\mu A_\mu = j_\nu. \tag{3.1}$$

This approach is based on equation (2.8) which after appropriate normalisation of the current $j_\mu(x) = eqj^0_\mu(x)$ has the form

$$\Box \partial^\mu A_\mu = -2j^0_\mu \partial^\mu S. \tag{3.2}$$

Equation (2.2) has the priority to Maxwell's equations that the differential operator $\Box\partial_\mu$ is not degenerate.

Let us assume that we have managed to quantise equation (3.2) in Heisenberg representation provided the field satisfies the equation

$$\Box^2 S(x) = 0. \tag{3.3}$$

There are many papers (see e.g. Zwanziger 1978, Sotkov *et al* 1979, Mintchev 1980) devoted to the quantisation of the field S. Here we shall write only the commutator

$$[S^-(x), S^+(y)] = -\mathrm{i}\lambda E^-(x-y) \tag{3.4}$$

where $S^\pm(x)$ denote the creation and annihilation operators respectively, λ is an arbitrary constant and

$$E^\pm(x) = \pm \mathrm{i}(4\pi)^{-2} \ln(-\mu^2 x^2 \mp \mathrm{i}0x_0). \tag{3.5}$$

The full commutation function $E(x)$ has the form

$$E(x) = E^+(x) + E^-(x) = (8\pi)^{-1} \varepsilon(x_0)\theta(x^2) \tag{3.6}$$

and it also follows that

$$[S(x), S(y)] = -\mathrm{i}\lambda E(x-y). \tag{3.7}$$

The field $S(x)$ has the right transformation properties under the conformal group in the sense that the transformation (2.4) preserves both equation (3.3) and the commutator (3.4) (see Sotkov *et al* 1979, Sotkov and Stoyanov 1980).

As usual, the problem of the quantisation of equation (3.2) reduces to consistent determination of the operators $j^0_\mu(x)$ and $A_\mu(x)$. If $j^0_\mu(x)$ is not an external classical current it can be expressed by the electron–positron fields $\psi(x)$ and $\bar\psi(x)$ which satisfy the Dirac equation with interaction. This would be the standard quantisation procedure for interacting fields if equation (3.2) were not a third-order differential equation.

We assume further that we have solved the problem for a quantisation of equation (3.2). It would mean that besides the operators A_μ, j^0_μ, S and ψ we have constructed the space $\mathscr{H}$, where these operators act. Let $|\phi\rangle$ denote the vector of $\mathscr{H}$ and let us write equation (3.2) for the matrix elements of the corresponding operators:

$$\Box\partial^{\mu}\langle\phi_1|A_{\mu}(x)|\phi_2\rangle=-2\int_{|\phi\rangle\in\mathscr{H}}\langle\phi_1|j^0_{\mu}(x)|\phi\rangle\langle\phi|\partial^{\mu}S(x)|\phi_2\rangle \tag{3.8}$$

The symbol $\int_{|\phi\rangle\in\mathscr{H}}$ stands for summation over the vectors of the relevant full basis in $\mathscr{H}$.

Evidently equation (3.8) is of the form (2.8) and, therefore, the results obtained in § 2 are also valid for it. In particular, we shall illustrate once again the validity of property 1 . For this purpose let us write the transformations (2.2), (2.4) and (2.6) for the matrix elements of the operators A_{μ}, j^0_{μ} and S. The infinitesimal form of these transformation laws is

$$T'(x;\phi_i)=T(x;\phi_i)+\delta T$$

$$\delta\langle\phi_1|A_{\mu}(x)|\phi_2\rangle=\delta\alpha^{\rho}[(x^2\delta^{\sigma}_{\rho}-2x^{\sigma}x_{\rho})\delta^{\nu}_{\mu}\partial_{\sigma}-2g_{\mu\rho}x^{\nu}-2\delta^{\nu}_{\mu}x_{\rho}+2\delta^{\nu}_{\rho}x_{\mu}]\times\langle\phi_1|A_{\nu}(x)|\phi_2\rangle+\partial_{\mu}\langle\phi_1|C|\phi_2\rangle \tag{3.9}$$

$$\delta\langle\phi_1|j^0_{\mu}(x)|\phi\rangle=\delta\alpha^{\rho}[(x^2\delta^{\sigma}_{\rho}-2x_{\rho}x^{\sigma})\delta^{\nu}_{\mu}\partial_{\sigma}-2g_{\rho\mu}x^{\nu}+2x_{\mu}\delta^{\nu}_{\rho}-6x_{\rho}\delta^{\nu}_{\mu}]\langle\phi_1|j^0_{\mu}|\phi\rangle \tag{3.10}$$

$$\delta\langle\phi|\partial_{\mu}S(x)|\phi_2\rangle=\delta\alpha^{\rho}\{[(x^2\delta^{\sigma}_{\rho}-2x_{\rho}x^{\sigma})\delta^{\nu}_{\mu}\partial_{\sigma}-2g_{\mu\rho}x^{\nu}-2\delta^{\nu}_{\mu}x_{\rho}+2\delta^{\nu}_{\rho}x_{\mu}]\times\langle\phi|\partial_{\nu}S(x)|\phi_2\rangle-2eg_{\mu\rho}\langle\phi|q|\phi_2\rangle\} \tag{3.11}$$

(the properties of q and C as operators are investigated in Sotkov *et al* (1979) and Sotkov and Stoyanov (1980)). We then obtain for conformal-symmetric solutions of (3.8) an identity analogous to (2.11):

$$[(x^2\delta^{\nu}_{\mu}-2x_{\mu}x^{\nu})\partial_{\nu}-8x_{\mu}]\Box\partial^{\rho}\langle\phi_1|A_{\rho}(x)|\phi_2\rangle+4\partial^{\rho}\langle\phi_1|F_{\rho\mu}(x)|\phi_2\rangle$$
$$=-2[(x^2\delta^{\nu}_{\mu}-2x_{\mu}x^{\nu})\partial_{\nu}-8x_{\mu}]\int_{|\phi\rangle\in\mathscr{H}}\langle\phi_1|j^0_{\rho}(x)|\phi\rangle\langle\phi|\partial^{\rho}S(x)|\phi_2\rangle$$
$$+\int_{|\phi\rangle\in\mathscr{H}}\langle\phi_1|j^0_{\mu}|\phi\rangle\langle\phi|eq|\phi_2\rangle. \tag{3.12}$$

Hence we conclude that the conformal-symmetric matrix elements of the operator $A_{\mu}(x)$ (i.e. those of them which satisfy (3.12)) are also solutions of Maxwell's equations:

$$\partial^{\rho}\langle\phi_1|F_{\rho\mu}(x)|\phi_2\rangle=\langle\phi_1|eqj^0_{\mu}(x)|\phi_2\rangle. \tag{3.13}$$

Thus, the conformal-symmetric property of the matrix elments is a criterion that they satisfy (3.13).

We note that (3.2) (and also (3.8)) determines a very large quantum system, whose state space $\mathscr{H}$ includes not only QED. This statement follows from the fact that solutions of equation (3.8) exist which are not conformal symmetric.

Let $\mathscr{H}^0\subset\mathscr{H}$ be the subspace of vectors $|\phi\rangle$ such that the matrix elements $\langle\phi_1|A_{\mu}(x)|\phi_2\rangle$, $\phi_i\in\mathscr{H}$ have the conformal symmetric property. It is evident that these matrix elements satisfy (3.13) and hence the space $\mathscr{H}^0$ contains the physical photon states of QED. The transformation (3.9) for the conformal symmetric matrix elements induces in $\mathscr{H}^0$ a representation of the conformal algebra.

In this way the quantisation of equation (3.2) and the separation of the subspace $\mathscr{H}^0 \subset \mathscr{H}$ determine a conformal-invariant procedure for quantisation of the electromagnetic field. The further construction of the physical state space $\mathscr{H}_{\text{ph}} \subset \mathscr{H}^0$ depends on the concrete structure of $\mathscr{H}^0$ and this will be demonstrated in § 4.

4. Free electromagnetic field

Equation (3.2), in the case of the free electromagnetic field, takes the form

$$\Box \partial^\mu A_\mu(x) = 0. \tag{4.1}$$

In order to construct the quantum theory of A_μ we must find the general form of the commutation function satisfying (4.1) with the standard initial conditions. Here we propose a particular form of scale and Poincaré invariant commutation relation with the necessary gauge freedom (2.13):

$$[A_\mu(x), A_\nu(y)] = -\mathrm{i}g_{\mu\nu}D(x-y) + \mathrm{i}\varkappa\partial_\mu\partial_\nu E(x-y) \equiv \Delta_{\mu\nu}(x-y) \tag{4.2}$$

$$[A_\mu(x), S(y)] = \mathrm{i}\tau\partial_\mu E(x-y) \equiv \Delta_\mu(x-y). \tag{4.3}$$

Here $\varkappa$ and τ are normalisation constants and $D(x)$ is the commutation function of the massless scalar field:

$$D(x) = (2\pi)^{-1}\varepsilon(x_0)\delta(x^2).$$

The functions $D(x)$ and $E(x)$ (and also their frequency parts) are related by the equalities

$$\Box E(x) = D(x) \qquad \partial_\mu E(x) = \tfrac{1}{2}x_\mu D(x). \tag{4.4}$$

The commutation relations (4.2) and (4.3) for the frequency parts $A_\mu^\pm$, $S^\pm$ have the form

$$[A_\mu^-(x), A_\nu^+(x)] = -\mathrm{i}g_{\mu\nu}D^-(x-y) + \mathrm{i}\varkappa\partial_\mu\partial_\nu E^-(x-y) \tag{4.5}$$

$$[A_\mu^-(x), S^+(y)] = \mathrm{i}\tau\partial_\mu E^-(x-y) \tag{4.6}$$

and all other commutators are equal to zero.

We shall realise the operators A_μ and S satisfying equations (4.1) and (3.3) in the Fock space $\mathscr{H}$. Since the results of this section can be easily generalised for the n-particle states we consider here only the one-particle state space $\mathscr{H}_{(1)} \subset \mathscr{H}$.

Let us denote with $|0\rangle$ the common vacuum vector of A_μ and S:

$$A_\mu^-(x)|0\rangle = 0 = S^-(x)|0\rangle.$$

Then the general form of the one-particle states is the following:

$$|\phi_{(1)}\rangle = \int \varphi^\mu(x)A_\mu^+(x)|0\rangle\,\mathrm{d}^4x + \int \chi(x)S^+(x)|0\rangle\,\mathrm{d}^4x \tag{4.7}$$

where $\varphi_\mu(x)$ and $\chi(x)$ belong to the space $S(R_4)$ of the infinitely differentiable functions of four arguments which tend to zero more quickly than any powers of $1/|x|$ when $|x|$ tends to infinity. The vectors (4.7) form the subspace of one-particle states $\mathscr{H}_{(1)}$. It is evident from (4.7) that any vector $|\phi_{(1)}\rangle \in \mathscr{H}_{(1)}$ is described by five functions $\{\varphi_\mu, \chi\}$. The commutators (4.5), (4.6) and (3.4) define the scalar product in the one-particle Fock space $\mathscr{H}_{(1)}$ which in terms of φ_μ and χ has the form

$$\langle\psi_{(1)}|\phi_{(1)}\rangle=\int \mathrm{d}^4x\ \mathrm{d}^4y\ (\bar{\psi}^\mu(x),\bar{\sigma}(x))\begin{pmatrix}\Delta^-_{\mu\nu}(x-y) & \Delta^-_\mu(x-y)\\ \Delta^-_\nu(x-y) & -\mathrm{i}\lambda E^-(x-y)\end{pmatrix}\begin{pmatrix}\varphi^\nu(y)\\ \chi(y)\end{pmatrix}. \tag{4.8}$$

Obviously the form (4.8) is not positive definite.

Let us consider the unique non-zero matrix element of A_μ taken over the states from $\mathscr{H}_{(1)}$:

$$\begin{aligned}B_\mu(x;\phi_{(1)})&=\langle 0|A^-_\mu(x)|\phi_{(1)}\rangle\\ &=-\mathrm{i}\int\varphi_\mu(y)D^-(x-y)\,\mathrm{d}^4y+\mathrm{i}\varkappa\int\varphi^\nu(y)\partial_\mu\partial_\nu E^-(x-y)\,\mathrm{d}^4y\\ &\quad+\mathrm{i}\tau\int\chi(y)\partial_\nu E^-(x-y)\,\mathrm{d}^4y.\end{aligned} \tag{4.9}$$

It follows immediately from the explicit form of B_μ that these matrix elements satisfy equation (4.1) for any $|\phi_{(1)}\rangle\in\mathscr{H}_{(1)}$. As was noted in § 3 we should separate from the obtained solutions of equation (4.1) those of them which are conformal symmetric. This means that we separate from equation (4.9) the solutions which satisfy the equation

$$\Box\partial^\mu[(\partial x'^\nu/\partial x^\mu)B_\nu(x';\phi_{(1)})]=0.$$

The direct calculation leads to the following restriction for the functions $\varphi_\mu(x)$:

$$\partial^\mu\varphi_\mu(x)=0 \tag{4.10}$$

while $\chi(x)$ remains arbitrary. For simplicity, in order to work only with homogeneous transformations for the matrix elements, we impose one more restriction on φ_μ and χ:

$$\int\chi(x)\,\mathrm{d}^4x=0=\int\varphi_\mu(x)\,\mathrm{d}^4x \tag{4.11}$$

which means that we shall consider only the subspace $S_0(R_4)\subset S(R_4)$. The conditions (4.10) and (4.11) determine the subspace $\mathscr{H}^0_{(1)}\subset\mathscr{H}_{(1)}$ and it is easy to check that $B_\nu(x;\phi)$ for any $|\phi_{(1)}\rangle\in\mathscr{H}^0_{(1)}$ satisfy Maxwell's equations:

$$\Box B_\mu-\partial_\mu\partial^\nu B_\nu\equiv\langle 0|\partial^\nu F_{\nu\mu}(x)|\phi_{(1)}\rangle=0. \tag{4.12}$$

Therefore, the physical photon states belong to $\mathscr{H}^0_{(1)}$.

We shall now prove that in $\mathscr{H}^0_{(1)}$ a representation of the conformal algebra is realised. The requirement for $B_\mu(x;\phi_{(1)})$ to transform according to the law (2.6) has the infinitesimal form (3.9), where

$$\delta\langle 0|A_\mu(x)|\phi_{(1)}\rangle\equiv\langle 0|A_\mu(x)|\phi'_{(1)}\rangle-\langle 0|A_\mu(x)|\phi_{(1)}\rangle$$

$$|\phi'_{(1)}\rangle=\int\varphi'_\mu(x)A^+_\mu(x)|0\rangle\,\mathrm{d}^4x+\int\chi'(x)S^+(x)|0\rangle\,\mathrm{d}^4x.$$

Here $\varphi'_\mu(x)$ and $\chi'(x)$ are the transformed functions whose transformation laws we are looking for. Substituting the explicit form (4.9) of $B_\mu(x;\phi_{(1)})$ in equation (3.9) and taking into account the properties of the functions Δ_μ and $\Delta_{\mu\nu}$ and the conditions (4.10) and (4.11), we obtain for the special conformal transformations of the functions

$\{\phi_\mu(x), \chi(x)\}$

$$(\tau\chi)' = \tau\chi + \delta(\tau\chi) \qquad \varphi'_\mu(x) = \varphi_\mu(x) + \delta\varphi_\mu(x)$$

where

$$\delta(\tau\chi(y)) = [y^2(\alpha\partial) - 2(\alpha y)(y\partial)](\tau\chi(y)) - 8(\alpha y)(\tau\chi(y)) - 4(\alpha\varphi(y)) \quad (4.13)$$

$$\delta\varphi_\mu(y) = [y^2(\alpha\partial) - 2(\alpha y)(y\partial)]\varphi_\mu(y) - 6(\alpha y)\varphi_\mu(y) + 2y_\mu(\alpha\varphi) - 2\alpha_\mu(y\varphi). \quad (4.14)$$

It is easy to check that the conditions (4.10) and (4.11) are invariant under the transformations (4.13) and (4.14). Hence *the space $\mathscr{H}^0_{(1)}$ is conformal invariant.*

Remark 1. As can be seen from equation (4.13) the variation of the field $(\tau\chi(y))$ does not vanish even when $\tau = 0 = \varkappa$ (i.e. the Feynman gauge). In this case we have

$$\delta(\tau\chi(y)) = -4(\alpha\cdot\varphi(y))$$

which means that the field $S(x)$, absent in the Feynman gauge, appears here.

Remark 2. As is shown in Stoyanov and Sotkov (1981) the condition

$$\partial^\mu\tilde{\varphi}_\mu(x) = \Box R(x) \qquad R(x)\in S(R_1) \quad (4.15)$$

separates the space $\tilde{\mathscr{H}}^0_{(1)}$ larger than $\mathscr{H}^0_{(1)}$, i.e. $\mathscr{H}^0_{(1)} \subset \tilde{\mathscr{H}}^0_{(1)}$. This larger space is conformal invariant also and equations (4.12) are satisfied in it. The transformation laws of the vectors $|\tilde{\phi}_{(1)}\rangle \in \tilde{\mathscr{H}}^0_{(1)}$ (determined by the functions $\{\tilde{\varphi}_\mu, \tilde{\chi}\}$) which have been found in the paper quoted above can reduce to those given by (4.13) and (4.14) with the help of the transformations

$$\tilde{\varphi}_\mu = \varphi_\mu + \partial_\mu R \qquad \tilde{\chi} = \tau\chi - (\varkappa - 1)\Box R. \quad (4.16)$$

These transformations map the space $\tilde{\mathscr{H}}^0_{(1)}$ into $\mathscr{H}^0_{(1)}$ and they reduce the condition (4.15) to (4.10). The vectors $|\tilde{R}\rangle \in \mathscr{H}^R_{(1)}$ determined by the functions $\{\partial_\mu R, \Box R\}$ have zero length. It is easy to see that the space $\mathscr{H}^0_{(1)}$ is isomorphic to the factor space $\tilde{\mathscr{H}}^0_{(1)}/\mathscr{H}^R_{(1)}$.

5. Physical photon states

The subspace $\mathscr{H}^0_{(1)}$ constructed in § 4 does not satisfy all requirements for a physical space. It is easy to see that the scalar product (4.8) which in $\mathscr{H}^0_{(1)}$ has the form

$$\langle\psi^0_{(1)}|\phi^0_{(1)}\rangle = -\mathrm{i}\int \bar{\varphi}^\mu_1(x)\varphi_{2\mu}(y)D^-(x-y)\,\mathrm{d}^4x\,\mathrm{d}^4y - \mathrm{i}\lambda\int \bar{\chi}_1(x)\chi_2(y)E^-(x-y)\,\mathrm{d}^4x\,\mathrm{d}^4y \quad (5.1)$$

is not positive definite.

We shall consider here two mutually orthogonal subspaces of $\mathscr{H}^0_{(1)}$. The first one, $\mathscr{H}^0_\varphi$, consists of the vectors $|\varphi\rangle$ determined by the functions $\{\varphi_\mu, 0\}$. The scalar squares of these vectors are non-negative. The second space $\mathscr{H}^0_\chi$ consists of the vectors $|\chi\rangle$ characterised by the functions $\{0, \chi\}$. The scalar product (5.1) in the space $\mathscr{H}^0_\chi$ is indefinite (see Zwanziger 1978, Sotkov *et al* 1979, Mintchev 1980). Starting from the laws (4.12) and (4.13) it is easy to check that the space $\mathscr{H}^0_\chi$ is conformal invariant. At the same time the special conformal transformations bring the vectors $|\varphi\rangle$ out of the space $\mathscr{H}^0_\varphi$. This means that the representation given by (4.13) and (4.14) is non-decomposable.

An important property of the space $\mathscr{H}^0_\chi$ is that its vectors $|\chi\rangle$ are annihilated by the negative frequency part of the Maxwell tensor

$$F^-_{\mu\nu}|\chi\rangle = 0. \tag{5.2}$$

Obviously this property shows that the vectors from $\mathscr{H}^0_\chi$ cannot be considered as physical photon states even if they have positive scalar square. Then we define the subspace of one photon with the non-negative scalar square as the factor space

$$\mathscr{H}'_{(1)} = \mathscr{H}^0_{(1)}/\mathscr{H}^0_\chi.$$

In order to obtain the one-photon physical space $\mathscr{H}^{\text{ph}}_{(1)}$ we should factorise $\mathscr{H}'_{(1)}$ by the subspace $\mathscr{H}''_{(1)}$ of the zero length vectors, i.e. satisfying the condition $\langle\phi_{(1)}|\phi_{(1)}\rangle = 0$ where

$$\langle\phi_{(1)}|\phi_{(1)}\rangle = -\mathrm{i}\int \bar{\varphi}^\mu(x)\varphi_\mu(y)D^-(x-y)\,\mathrm{d}^4x\,\mathrm{d}^4y. \tag{5.3}$$

Evidently the expression (5.3) determines a norm in the space $\mathscr{H}'_{(1)}$. In this way the physical one-photon state space is

$$\mathscr{H}^{\text{ph}}_{(1)} = \mathscr{H}'_{(1)}/\mathscr{H}''_{(1)}.$$

Since the norm (5.3) in $\mathscr{H}'_{(1)}$ is positive definite and conformal invariant it is evident that: *in the physical space $\mathscr{H}^{\text{ph}}_{(1)}$ a unitary irreducible representation of the conformal group is realised.*

The n-particle state space and the full physical space are constructed by the standard procedure from the one-particle space (see e.g. Strocchi and Wightman (1974)) and we do not discuss their properties here.

6. Concluding remarks

In QED with interactions usually one uses the following equations for the electromagnetic field (the so-called ξ gauge):

$$\Box A_\mu - \xi\partial_\mu\partial^\nu A_\nu = j_\mu \tag{6.1}$$

$$\xi = \text{constant} \neq 1 \qquad \partial^\mu j_\mu = 0. \tag{6.2}$$

It follows immediately from (6.1) that all its solutions (even with different ξ and j_μ) are solutions also of equation (4.1). We shall show (if the quantisation preserves condition (6.2)) that the space $\mathscr{H}^{\text{ph}}$ (and also $\mathscr{H}^0$) determined in § 5 has no common elements (except the zero) with the state space constructed on the basis of equation (6.1) and satisfying the Gupta–Bleuler condition

$$\langle\phi^1_{\mathrm{G}}|\partial^\mu A_\mu|\phi^2_{\mathrm{G}}\rangle = 0. \tag{6.3}$$

Indeed, due to equation (6.1) in the space $\mathscr{H}^{\text{ph}}$ (and $\mathscr{H}^0$) only the free Maxwell's equations are satisfied whereas in the Gupta–Bleuler state space with interaction we have

$$\langle\phi^1_{\mathrm{G}}|\Box A_\mu - j_\mu|\phi^2_{\mathrm{G}}\rangle = 0. \tag{6.4}$$

Since the free Gupta-Bleuler state space ($j_\mu = 0$) is a subspace of our space $\mathscr{H}^{\text{ph}}$ (or $\mathscr{H}^0$) we have proved the following statement: *the Gupta–Bleuler physical spaces with and without interaction cannot have common elements* (*with the exception of the zero vector*).

On the other hand, in our formulation of QED on the basis of equations (3.1) and (3.2) such a statement is not valid. But it turns out that in addition the physical state space with interaction is also conformal invariant.

We note that equation (3.2) can arise as a consequence from an equation of the form (6.1) but with another current:

$$\Box A_\mu - \xi \partial_\mu \partial^\nu A_\nu = S(x) j_\mu(x) \qquad \partial^\mu j_\mu = 0 \qquad \xi \neq 1. \tag{6.5}$$

Finally we remark also that the commutator function (4.2) chosen in this paper is a particular solution of equation (4.1) and evidently it is not the only one. Therefore, in the free field case it is possible on the basis of a more general commutation function to construct a quantum system leading to the state space larger than $\mathscr{H}^0$.

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